MATB24 Final Review

① Vector Space

def: ① closed under addition $\forall v,w \in \mathbb{V},\ v+w \in \mathbb{V}$

② closed under scalar multiplication $\forall v \in \mathbb{V}, r \in \mathbb{R},\ rv \in \mathbb{V}$

③ A_1-A_4 S_1-S_4

- $\{\vec{v}_1,\dots,\vec{v}_n\}$ is a basis of $\mathbb{V}$ if every $v \in \mathbb{V}$ can be expressed uniquely as a linear combination of $\{\vec{v}_1,\dots,\vec{v}_n\}$

② Subspace

def: ① non-empty

② closed under addition $\forall v,w \in \mathbb{V},\ v+w \in \mathbb{V}$

③ closed under scalar multiplication $\forall v \in \mathbb{V}, r \in \mathbb{R},\ rv \in \mathbb{V}$

- if $\mathbb{V}$ has a dimension V, $x \in \mathbb{V}$ be L.I and $|x| = X$, $\text{span}(y) = \mathbb{V}$, $|y| = y$. then $x \le v \le y$

- $\text{Span}(v_1,\dots,v_n) = \text{span}(w_1,\dots,w_m) \iff v_1,\dots,v_n \in \text{span}(w_1,\dots,w_m)$ and $w_1,\dots,w_m \in \text{span}(v_1,\dots,v_n$

③ Linear Transformation

Def: ① $T(\vec{v}_1+\vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

② $T(r\vec{v}) = rT(\vec{v})$

injective: $T(v) = T(w) \iff v = w$ / $\text{Ker}(T) = 0$

surjective: $\forall w \in \mathbb{W}, \exists v \in \mathbb{V}$ s.t. $T(v) = w$ / $\text{Im}(T) = \mathbb{W}$

④ Change of Basis

$T_A : [V]_\varepsilon \mapsto [V]_A \quad T_B : [V]_\varepsilon \mapsto [V]_B$

$T_{A\to B} : [V]_A \mapsto [V]_B$, $C_{A\mapsto B}[V]_A = [V]_B$, where $C_{A\mapsto B} = \begin{bmatrix} | & | & & | \\ [a_1]_B & [a_2]_B & \cdots & [a_n]_B \\ | & | & & | \end{bmatrix}$

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dl, "T_A"'] \arrow[dr, "T_B"] & \\ \mathbb{R}^n \arrow[rr, "T_{A\to B}"] & & \mathbb{R}^n \end{tikzcd}$$

$\bar{T}: [V]_A \to [T(v)]_B$

$[T]_{A,B}[V]_A = [T(\vec{v})]_B$, where $[T]_{A,B} = \begin{bmatrix} | & | & & | \\ [T(a_1)]_B & [T(a_2)]_B & \cdots & [T(a_n)]_B \\ | & | & & | \end{bmatrix}$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "T_A"'] & W \arrow[d, "T_B"] \\ \mathbb{R}^n \arrow[r, "\bar{T}"] & \mathbb{R}^m \end{tikzcd}$$

$[T]_{A,A}\,[V]_A = [T(v)]_A$, where $[T]_{A,A} = \begin{bmatrix} | & | & & | \\ [T(a_1)]_A & [T(a_2)]_A & \cdots & [T(a_n)]_A \\ | & | & & | \end{bmatrix}$

⑤ Inner Product Space

Def: ① $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle \; \forall \vec{v}, \vec{w} \in V$

② $\langle \vec{u} + \lambda\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \lambda \langle \vec{v}, \vec{w} \rangle$

③ $\langle \vec{v}, \vec{v} \rangle \geq 0 \; \forall \vec{v} \in V$ and $\langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = \vec{0}_V$

- $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$
- $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
- $\theta = \arccos\left(\frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|}\right)$
- $\vec{v} = \vec{v}_{\vec{w}} + \vec{v}_{\vec{w}^\perp}$
  $= \mathrm{proj}_{\vec{w}} \vec{v} + v - \mathrm{proj}_{\vec{w}} \vec{v}$
  $= \left(\frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w}\right) + \left(\vec{v} - \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w}\right)$

Gram Schidt process

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of V, find $\{\vec{u}_1, \dots, \vec{u}_n\}$ which is an orthonormal basis

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$\vec{y}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$, $\vec{u}_2 = \frac{\vec{y}_2}{\|\vec{y}_2\|}$

$\vec{y}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$, $\vec{u}_3 = \frac{\vec{y}_3}{\|\vec{y}_3\|}$

note: $\mathrm{proj}_V \vec{x} = (x \cdot v_1)v_1 + (x \cdot v_2)v_2 + \dots + (x \cdot v_n)v_n$, if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthnal basis

- if $V = \mathrm{sp}\{v_1, \dots, v_n\}$, then basis of $V^\perp$ is $\mathrm{null}(A^T)$, where $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

⑥ Orthogonal Linear Transformation

Def: $T: V \to V$ is orthogonal if $\langle T(v), T(w) \rangle = \langle v, w \rangle$, $\forall v, w \in V$

Properties: ① $\|T(\vec{v})\| = \|\vec{v}\|$ presene length

② presene angle

③ $\|T(\vec{v})-T(\vec{w})\| = \|\vec{v}-\vec{w}\|$ presene distance

④ the standard matrix of T 's columns are orthonormal and $A^TA = I_n$

Projection matrix: $Proj_{\vec{w}}\vec{v} = P\vec{v}$ where $\{\vec{b_1},...,\vec{b_n}\}$ is a orthogonal basis of w and $A = [\vec{b_1}\ \vec{b_2} \cdots \vec{b_n}]$, $P = A(A^TA)^{-1}A^T$

Property: $P^2 = P$ (idempotence)

$P = P^T$ (symmetry)

QR: $A = QR$ where col(A) is LI, Q is orthonormal matrix. R is upper-triangle matrix

Q = col(A) been Gram-Schmidted and $R = Q^TA$

Least square method

$A\vec{y} = Proj_{col(A)}\vec{b} \Rightarrow A\vec{y} = A(A^TA)^{-1}A^T\vec{b} \Rightarrow \vec{y} = (A^TA)^{-1}A^T\vec{b}$ $\rightarrow$ $(A^TA)^{-1}A^T\vec{b} = ((QR)^TQR)^{-1}(QR)^T\vec{b} = R^{-1}Q^T\vec{b}$

7 Bilinear Transformation

Def: $f(r\vec{v_1}+\vec{v_2}, \vec{w}) = rf(\vec{v_1},\vec{w}) + f(\vec{v_2},\vec{w})$

$f(\vec{v}, r\vec{w_1}+\vec{w_2}) = rf(\vec{v},\vec{w_1}) + f(\vec{v},\vec{w_2})$

Multilinear $\approx$ Bilinear but multiple inputs instead of 2.

- alternation (when two input the same): $f(\vec{v_1},...,\vec{v_i},...,\vec{v_i},...,\vec{v_n}) = 0$
- skew-symmetric (switch two inputs' position): $f(\vec{v_1},...,\vec{v_i},...,\vec{v_j},...,\vec{v_n}) = -f(\vec{v_1},...,\vec{v_j},...,\vec{v_i},...,\vec{v_n})$

alternating $\Leftrightarrow$ Skew-symmetric

determinant map from $(\mathbb{R}^n)^n \mapsto \mathbb{R}$ is $D(e_1,...,e_n) = 1$

A n-box made with n linearly independent vectors in $\mathbb{R}^m$ is $r_1v_1+r_2v_2+...+r_nv_n$ $r_i \in [0,1]$

$\det(T(V)) = |\det(T)|\det(V)|$

when A is not n×n, $= \sqrt{\det(A^TA)}$ apply both for det(T) and det(V)

$\det(T) = \det([T]_B)$ where B is any basis for V if $T: V \mapsto V$

A is diagonalizable $\Leftrightarrow$ $\exists$ diagonal D, invertible P s.t. $A = PDP^{-1}$

$T: V \mapsto V$

$T(\vec{v}) = \lambda\vec{v}$, $\vec{v}$ is called the eigenvector of T and λ is the corresponding eigenvalue

$T: V \mapsto V$ is diagonalizable if $\exists$ a basis B for V s.t. $[T]_B$ is diagonal.

(8) Complex System

$z=(a+bi), w=(c+di), zw=(ac-bd)+(ad+bc)i$

Def of complex vector space V: ① nonempty
② $+: V\times V \mapsto V$ and $\cdot : \mathbb{C}\times V \mapsto V$
③ $A_1 \sim A_4, S_1 \sim S_4$

$\langle z,w\rangle_{\mathbb{C}} = \bar{z}\cdot w$

Property of Euclidean Inner Product over $\mathbb{C}$

① $\langle z+w,y\rangle_{\mathbb{C}} = \langle z,y\rangle_{\mathbb{C}} + \langle w,y\rangle_{\mathbb{C}}$, $\langle y,z+w\rangle_{\mathbb{C}} = \langle y,z\rangle_{\mathbb{C}} + \langle y,w\rangle_{\mathbb{C}}$ ② $\langle rz,w\rangle_{\mathbb{C}} = \bar{r}\langle z,w\rangle_{\mathbb{C}}$ $\langle z,rw\rangle_{\mathbb{C}} = r\langle z,w\rangle$ ③ $\langle z,w\rangle_{\mathbb{C}} = \overline{\langle w,z\rangle}_{\mathbb{C}}$ ④ $\langle z,z\rangle_{\mathbb{C}} \geq 0$ and $\langle z,z\rangle_{\mathbb{C}} = 0$ iff $z=0$

Def of hermitian inner product is $\langle\ ,\ \rangle_{\mathbb{C}}: V\times V \mapsto \mathbb{C}$ s.t satisfies ①②③④ and $\langle z,w\rangle_{\mathbb{C}} = \bar{z}^T w$

Property of $*$

① $(A+B)^* = A^* + B^*$

② $(rA)^* = \bar{r}A^*$

③ $(AB)^* = B^*A^*$

$\mathbb{R}$	$\mathbb{C}$
$\langle\ ,\ \rangle$	$\langle\ ,\ \rangle_{\mathbb{C}}$
$v\cdot w = v^T w$	$\langle v,w\rangle = v^* w$
$A=A^T$ sym	$A=A^*$ Hermitian
$U^TU=UU^T=I_n$	$UU^*=U^*U=I_n$
orthogonal	unitary

Def $A\in M_{nn}(\mathbb{C})$ is called Hermitian if $A=A^*$

Def $A\in M_{nn}(\mathbb{C})$ is called unitary if $A^*A=AA^*=I_n$

columns of unitary matrix are orthonormal basis (just like orthogonal matrix, so $U^*U=I_n$)

Fundamental Thm of Algebra: Any polynomial of degree n over $\mathbb{C}$ has n complex vectors

Def $A\in M_{nn}(\mathbb{R})$ is called orthogonally diagonalizable if $\exists$ diagonal D and orthogonal U, s.t $A=UDU^T$

Def $T:\mathbb{R}^n \mapsto \mathbb{R}^n$ is orthogonally diagonalizable if $\mathbb{R}^n$ has an orthonormal eigenbasis of T

Def $A\in M_{nn}(\mathbb{C})$ is called unitary diagonalizable if $\exists$ a diagonal D, and a unitary U s.t $A=UDU^*$

Def $T:\mathbb{C}^n \mapsto \mathbb{C}^n$ is called unitary diagonalizable if $\mathbb{C}^n$ has an orthonormal eigenbasis of T.

Spectral Thm (for complex matrices)

$A \in M_n(\mathbb{C})$. If A is Hermitian then A is U.D. And all eigenvalues of A are real numbers.
(i.e $A = A^*$ then $\exists$ unitary U, $\exists$ diagonal D, $D \in M_n(\mathbb{R})$. s.t $A = UDU^*$)

Spectral Thm (for real matrices)

$A \in M_n(\mathbb{R})$. If A is sym. then A is O.D
(i.e $A = A^T$ then $\exists$ orthogonal U and $\exists$ diagonal D. $U, D \in M_n(\mathbb{R})$ s.t $A = UDU^T$)

Def $A, B \in M_n(\mathbb{C})$, A is unitary equivalent to B if $\exists$ unitary matrix U s.t $A = UBU^*$

$A \underset{U.E}{\sim} B$

thm A is U.D $\iff$ A U.E to a diagonal matrix

Def A matrix $A \in M_n(\mathbb{C})$, is called normal if $AA^* = A^*A$

Thm $A \in M_n(\mathbb{C})$, A is U.D $\iff$ A is normal

Lemma: If A is normal and $A \underset{U.E}{\sim} B$, then B is normal

⑨ Jordan Block

Def A $n \times n$ matrix of the for $\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$ is called Jordan block

Def A matrix $A = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{bmatrix}$, where J_i are Jordan block is called to be in Jordan Canonical form